UDC 62-50: 534

OPTIMAL CONTROL OF THE MOTION OF A QUASILINEAR OSCILLATORY SYSTEM BY SMALL FORCES

PMM Vol.39, № 6, 1975, pp.995-1005

L. D. AKULENKO
(Moscow)

(Received January 10, 1975)

We solve certain optimal control problems for the motion of a single-frequency oscillatory system which in the unperturbed state consists of an arbitrary number of oscillating elements. The solution is performed in the first approximation with respect to a small parameter ε . We assume that the frequency depends upon slow time, while the control goes only into the perturbing terms, so that the system is formally weakly controllable [1]. But since the time interval over which the process evolves is a quantity $\sim 1/\varepsilon$, all the controlled quantities are able to vary substantially [2, 3], i.e. we investigate the case, interesting in practice, of small but protracted control forces. As mechanical examples we calculate some optimal control problems for the oscillations of systems of the plane oscillator type, etc.

1. Statement of the optimal control problems and the construction of the averaged boundary-value problems of the maximum principle. We consider a quasilinear mechanical system of the form

$$\mu(\tau)r^{\cdot\cdot} + k(\tau)r = \varepsilon f(\tau, r, r^{\cdot}, u) + F(\tau) \qquad (1.1)$$
$$r(t_0) = r_0, \; r^{\cdot}(t_0) = r_0^{\cdot}$$

decomposing into n oscillating elements when $\varepsilon = 0$. Here $r = (x_1, \ldots, x_n)$ and $r^{\cdot} = dr / dt$ are generalized coordinates, $t \geqslant t_0$ is time, $\tau = \varepsilon t + \text{const}$ is "slow time", $\varepsilon \in [0, \varepsilon_0]$ is a small parameter, $\varepsilon > 0$, $u = (u_1, \ldots, u_m)$ is the control vector, $u \in U$, U is some fixed convex set, t_0, r_0, $r_0^{\cdot}$ are the initial data. The matrix-valued functions μ, the "mass", and k, the "rigidity factor", are assumed diagonal and strictly positive. All the functions are taken to be sufficiently smooth in the argument ranges being examined.

Formally system (1.1) is weakly controllable [1] since when $\varepsilon = 0$ there is no control

$$\mu(\tau)r^{\cdot\cdot} + k(\tau)r = F(\tau), \quad \tau = \text{const} \qquad (1.2)$$

Using the solution of system (1.2) we reduce (1.1) to a standard form, more convenient for investigation, by means of the relations

$$r = a\sin\psi + b\cos\psi + k^{-1}F, \quad r^{\cdot} = \nu(a\cos\psi - b\sin\psi) \qquad (1.3)$$
$$\nu^2 = \mu^{-1}k, \quad \psi = \int_{t_0}^{t} \nu(\tau_1)\, dt_1$$

Here a and b are new (slow) variables, ν^2 is a diagonal matrix of the squares of the natural frequencies, while an expression such as $a\sin\psi$ is a vector of the form $(a_1 \sin\psi_1, \ldots, a_n \sin\psi_n)$. By differentiating the relations (1.3), by virtue of the perturbed system (1.1), we obtain

$$\begin{aligned}
a^{\cdot} &= \varepsilon\nu^{-1}\mu^{-1}f\cos\psi - \varepsilon\nu^{-1}\nu'(a\cos\psi - b\sin\psi)\cos\psi - \\
&\quad \varepsilon(k^{-1}F)'\sin\psi \equiv \varepsilon A \\
b^{\cdot} &= -\varepsilon\nu^{-1}\mu^{-1}f\sin\psi + \varepsilon\nu^{-1}\nu'(a\cos\psi - b\sin\psi)\sin\psi - \\
&\quad \varepsilon(k^{-1}F)'\cos\psi \equiv \varepsilon B \\
\psi^{\cdot} &= \nu(\tau), \quad \psi(t_0) = 0 \\
(a(t_0) &= \nu^{-1}[\nu(r - k^{-1}F)\sin\psi + r^{\cdot}\cos\psi]_{t_0} \equiv a_0, \quad b(t_0) = \\
&\quad \nu^{-1}[\nu(r - k^{-1}F)\cos\psi - r^{\cdot}\sin\psi]_{t_0} \equiv b_0)
\end{aligned} \tag{1.4}$$

Here the prime denotes the derivative with respect to τ. In coordinate notation, for example, the first term of the upper equation has the form

$$\nu_i^{-1}\mu_i^{-1}f_i(\tau, a_j\sin\psi_j + b_j\cos\psi_j + k_j^{-1}F_j, \ \nu_j^{-1}(a_j\cos\psi_j - b_j\sin\psi_j), u_l)\cos\psi_i \quad (i, j = 1, 2, \ldots, n)$$

The right-hand side of system (1.4) is a complex multi-frequency function; the application of averaging methods to it leads to considerable difficulties evoked by resonances [4, 5]. Optimal control problems for an analogous system with $\nu = \text{const}$ were examined in [6]. For systems with variable frequencies the assumptions of absence or presence of resonance properties are rather artificial. Therefore, below we investigate system (1.4) under the condition that all the $\nu_i(\tau)$ coincide, namely, a single-frequency system. Then the right-hand side of (1.4) is periodic in ψ with period 2π. Certain specific mechanical problems of control that is optimum in the sense of various criteria of control by slow variables a and b for the single frequency systems are solved with application of the conical averaging method developed in [2, 3]. Here it is natural to assume that the functional and the terminal manifold do not depend strongly on time t or phase ψ.

We proceed to stating the optimal control problems for system (1.4) and to a brief formulation of the results of applying the averaging method. Let the performance index be

$$J = g(\tau, a, b)|_{\theta} + \varepsilon\int_{t_0}^{\theta} G(\tau, a, b, \psi, u)\,dt \to \min_{u\in U} \tag{1.5}$$

Here $\theta = T\ (T \sim 1/\varepsilon)$ is a specified quantity for problems with a fixed process termination instant and $\theta = t_1$ is an unknown quantity subject to determination for the time-optimal problem. Then, instant t_1 is chosen from the condition

$$M(\tau, a, b)|_{t_1} = 0, \quad M = (M_1, \ldots, M_L), \quad L \leqslant 2n \tag{1.6}$$

Suppose that the problems posed have solutions. Then the optimal control and trajectory satisfy the maximum principle [7]

$$H^* \equiv [-\varepsilon G + \varepsilon(pA) + \varepsilon(qB) + p_{\psi}\nu]^* = \max_{u\in U} H, \quad t \in [t_0, \theta] \tag{1.7}$$

Here H is the Hamilton function, the asterisk signifies that the functions are taken for $u = u^*(t)$, namely, the optimal control, and for the solution of system (1.4), corresponding to it, while the parentheses denote the scalar product. The variables p, q and p_{ψ} adjoint to a, b and ψ, respectively, satisfy the system

$$\begin{aligned}
&p^{\cdot} = -\varepsilon(\partial h/\partial a)^*, \quad q^{\cdot} = -\varepsilon(\partial h/\partial b)^*, \quad p_{\psi}^{\cdot} = -\varepsilon(\partial h/\partial\psi)^* \\
&H \equiv \varepsilon h + \nu p_{\psi}
\end{aligned} \tag{1.8}$$

and certain boundary conditions. For problems with fixed $\theta = T$

$$p(T) = -(\partial g / \partial a)_T, \quad q(T) = -(\partial g / \partial b)_T, \quad p_\psi(T) = 0 \qquad (1.9)$$

For the time optimal problem

$$p(t_1) = -\frac{\partial}{\partial a}(g + (\lambda M))_{t_1}, \quad q(t_1) = -\frac{\partial}{\partial b}(g + (\lambda M))_{t_1}, \quad p_\psi(t_1) = 0 \qquad (1.10)$$

Here $\lambda = (\lambda_1, \ldots, \lambda_L)$ is a parameter eliminated during the solving process. The equality

$$H^*|_{t_1} = \varepsilon\left(\frac{\partial g}{\partial \tau} + \left(\lambda \frac{\partial M}{\partial \tau}\right)\right)_{t_1} \qquad (1.11)$$

closes the system of boundary conditions.

If the maximum of H is achieved inside set U, and H is continuously differentiable in u_i, then

$$\partial H / \partial u_i = 0, \quad i = 1, \ldots, m \qquad (1.12)$$

These relations can be looked upon as equations relative to an unknown vector u. Thus, suppose that the control vector

$$u^* = V(\tau, a, b, \psi, p, q) \qquad (1.13)$$

has been found from condition (1.7) or, in particular, from Eqs. (1.12) and is a smooth function periodic in ψ. Substituting it into Eqs. (1.4) and (1.8), we obtain a boundary-value problem described by a standard system with rotating phase and a periodic right-hand side.

Let us write out the first-approximation boundary-value problems, using the procedures developed in [2, 3]. The corresponding equations do not contain the fast variable in the right-hand side and can be written in the slow time $s = \varepsilon t$

$$\frac{d\alpha}{ds} = A_0(\tau, \alpha, \beta, \xi, \eta), \quad \alpha(s_0) = a_0; \quad \frac{d\xi}{ds} = -\frac{\partial k_0}{\partial \alpha}(\tau, \alpha, \beta, \xi, \eta) \qquad (1.14)$$

$$\frac{d\beta}{ds} = B_0(\tau, \alpha, \beta, \xi, \eta), \quad \beta(s_0) = b_0; \quad \frac{d\eta}{ds} = -\frac{\partial k_0}{\partial \beta}(\tau, \alpha, \beta, \xi, \eta)$$

Here α, β, ξ and η are the averaged values of the variables a, b, p and q, respectively. The independent variable s varies on an interval of the order of unity: $s \in [s_0, \sigma]$, where $s_0 = \varepsilon t_0$, while $\sigma = \varepsilon T = S$ or $\sigma = \varepsilon t_1 \equiv s_1$. The right-hand side of system (1.14) is constructed by the averaged Hamiltonian k_0

$$k_0(\tau, \alpha, \beta, \xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} h^*(\tau, \alpha, \beta, \psi, \xi, \eta)\, d\psi \equiv \langle h^* \rangle \qquad (1.15)$$

$$h^*(\tau, a, b, \psi, p, q) = (pA(\tau, a, b, \psi, V)) + (qB(\tau, a, b, \psi, V)) - G(\tau, a, b, \psi, V)$$

We note that the identities

$$\left.\frac{\partial h}{\partial p}\right|_V = \frac{\partial h^*}{\partial p} = A, \quad \left.\frac{\partial h}{\partial a}\right|_V = \frac{\partial h^*}{\partial a} = \left(p\frac{\partial A}{\partial a}\right)^* + \left(q\frac{\partial B}{\partial a}\right)^* \qquad (1.16)$$

$$\left.\frac{\partial h}{\partial q}\right|_V = \frac{\partial h^*}{\partial q} = B, \quad \left.\frac{\partial h}{\partial b}\right|_V = \frac{\partial h^*}{\partial b} = \left(p\frac{\partial A}{\partial b}\right)^* + \left(q\frac{\partial B}{\partial b}\right)^*$$

are used here. From (1.16) follow

$$\frac{\partial k_0}{\partial \xi} = A_0, \quad \frac{\partial k_0}{\partial \alpha} = \left(\xi \frac{\partial A_0}{\partial \alpha}\right) + \left(\eta \frac{\partial B_0}{\partial \alpha}\right) \tag{1.17}$$
$$\frac{\partial k_0}{\partial \eta} = B_0, \quad \frac{\partial k_0}{\partial \beta} = \left(\xi \frac{\partial A_0}{\partial \beta}\right) + \left(\eta \frac{\partial B_0}{\partial \beta}\right)$$

Here and in (1.14) we have introduced the notation

$$\begin{Bmatrix} A_0 \\ B_0 \end{Bmatrix} (\tau, \alpha, \beta, \xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \begin{Bmatrix} A \\ B \end{Bmatrix} (\tau, \alpha, \beta, \psi, V)\, d\psi \tag{1.18}$$

The boundary conditions for the fixed-time problem are

$$\xi(S) = -(\partial g(\tau, \alpha, \beta) / \partial \alpha)_S, \qquad \eta(S) = -(\partial g(\tau, \alpha, \beta) / \partial \beta)_S \tag{1.19}$$

In the time-optimal problem the boundary conditions (1.19) are written analogously

$$\xi(s_1) = -\frac{\partial}{\partial \alpha}(g + (\lambda M))_{s_1}, \quad \eta(s_1) = -\frac{\partial}{\partial \beta}(g + (\lambda M))_{s_1} \tag{1.20}$$

$$g = g(\tau, \alpha, \beta), \qquad M = M(\tau, \alpha, \beta)$$

and analogously for (1.11)

$$k_0|_{s_1} = \left[\frac{\partial g}{\partial \tau} + \left(\lambda \frac{\partial M}{\partial \tau}\right)\right]_{s_1} \tag{1.21}$$

Thus, the uniquely derived solutions of the approximate boundary value problems provide an approximate solution for the input boundary value problems with an ε error and, also, for optimum control problems with an ε error with respect to the slow variables, while for the functional the error in the determination of the instant t_1 is of the order of unity.

The justification of the averaging method for smooth right-hand sides is contained in [4, 5]. However, in the case of a closed set U with a right-hand side of system (1.4), linear in u, which often holds in practice [8], the maximum of Hamiltonian (1.7) is achieved, as a rule, on the boundary of set U, i.e. the function V in (1.13) is piecewise-continuous and the number of discontinuities of the first kind in the right-hand side of the averaged system is of the order of $[1/\varepsilon]$. To justify the averaging method in this case we can apply the results of [6] wherein standard systems with discontinuous right-hand sides are investigated. We note that higher approximations can be constructed with the aid of the canonic averaging method developed in [2, 3].

2. Approximate solution of certain concrete optimal control problems. Everywhere below let function f be linear in u and let $m = n$, i.e. $f = f_0(\tau, r, r^{\cdot}) + u$. Let us first consider the problem of minimizing a quadratic functional of the form (1.5) with a specified instant T. Let $G = u^2 = \Sigma u_i^2$, set U be unbounded, while $g = \varkappa E$, $E = mr^{\cdot 2} / 2 + k(r - k^{-1}F)^2 / 2 = k(a^2 + b^2) / 2$. The quantity E has the sense of the energy of the oscillations; expressions of type $mr^{\cdot 2}$ are scalars $mr^{\cdot 2} = r^{\cdot T} m r^{\cdot} = m_1 r_1^{\cdot 2} + \ldots + m_n r_n^{\cdot 2}$.

By (1.12) and (1.13) we find: $u^* = (\nu^{-1}\mu^{-1} / 2)(p \cos\psi - q \sin\psi)$. Let us now compute the average value of Hamiltonian (1.15) determining the averaged system of equations

$$k_0 = (\nu^{-1}\mu^{-1})^2 (\xi^2 + \eta^2) / 8 - \nu^{-1}\nu' (\alpha\xi + \beta\eta) / 2 + \nu^{-1}\mu^{-1} (\xi f_{0c} - \eta f_{0s})$$

Here we have denoted

$$\begin{Bmatrix} f_{0c} \\ f_{0s} \end{Bmatrix} (\tau, \alpha, \beta) = \frac{1}{2\pi} \int_0^{2\pi} f_0 (\tau, \alpha \sin \psi + \beta \cos \psi + k^{-1} F,$$

$$\nu (\alpha \cos \psi - \beta \sin \psi)) \begin{Bmatrix} \cos \psi \\ \sin \psi \end{Bmatrix} d\psi$$

Let a small viscous friction $-\varepsilon \gamma r^{\cdot}$ act on system (1.1) ($\gamma(\tau)$ is a diagonal matrix with positive elements). Then $f_{0c} = -\gamma \alpha / 2$, and $f_{0s} = \gamma \beta / 2$. As a result we obtain a boundary-value problem with separable variables admitting an explicit integration. We write the solution using index notation

$$\begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} (s) = \begin{Bmatrix} a_{i0} \\ b_{i0} \end{Bmatrix} \exp \left(- \int_{s_0}^{s} \Lambda_i ds' \right) + \tag{2.1}$$

$$\begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix} (S) \int_{s_0}^{s} \Gamma_i (s') \exp \left(\int_S^{s'} \Lambda_i \, ds'' + \int_s^{s'} \Lambda_i ds'' \right) ds'$$

$$\begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix} (s) = \begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix} (S) \exp \left(\int_S^{s} \Lambda_i ds' \right), \quad s \in [s_0, S]$$

Here

$$\Lambda_i (s) = \nu_i^{-1} \nu_i' / 2 + \mu_i^{-1} \gamma_i, \quad \Gamma_i (s) = (\nu_i^{-1} \mu_i^{-1}) / 2, \quad \tau = s + \text{const} \tag{2.2}$$

$$\begin{Bmatrix} \xi_i \\ \eta_i \end{Bmatrix} (S) = - \varkappa k_i (S) \Big[1 + \varkappa k_i (S) \int_{s_0}^{S} \Gamma_i (s) \exp \left(2 \int_S^{s} \Lambda_i ds' \right) ds \Big]^{-1} \times$$

$$\begin{Bmatrix} a_{i0} \\ b_{i0} \end{Bmatrix} \exp \left(- \int_{s_0}^{S} \Lambda_i ds \right)$$

The expressions (2.1) and (2.2) obtained yield an approximate solution of the optimal control problem with an error of order ε for the time interval $t \in [t_0, T]$, where $T \sim 1 / \varepsilon$. We note that the value of E can be made as small as desired by increasing the constant $\varkappa > 0$; here the magnitude of the functional remains finite. Using these formulas we can obtain also an approximate optimal solution of the problem of "buildup", i.e. of increasing the energy of the oscillations, which corresponds to the case of $\varkappa < 0$.

Let us now consider the time-optimal problem with respect to energy

$$E |_{t_1} = E_1, \quad J = \chi \varepsilon t_1 + \varepsilon \int_{t_0}^{t_1} u^2 dt \to \min \quad (E, \chi > 0)$$

without constraints on the control. Suppose that assumptions analogous to the preceding example are fulfilled. Then the approximate optimal control has the form $u^* = \nu^{-1} \mu^{-1} (\xi \cos \psi - \eta \sin \psi) / 2 + O(\varepsilon)$, where ξ and η are computed from formulas (2.1) and (2.2) in which the parameters $\varkappa$ and σ are roots of the equations

$$\frac{1}{2} \sum_{i=1}^{n} k_i (\sigma) (a_{i0}^2 + b_{i0}^2) \exp \left(- \int_{s_0}^{\sigma} \Lambda_i ds \right) \Big[1 + \varkappa k_i (\sigma) \int_{s_0}^{\sigma} \Gamma_i (s) \times$$

$$\exp \left(2 \int_{\sigma}^{s} \Lambda_i ds_1 \right) ds \Big]^2 = E_1$$

$$\sum_{i=1}^{n}\left[\frac{1}{8\nu^2\mu_i^2}+\frac{1}{\varkappa k_i(\sigma)}\left(\frac{\nu'}{2\nu}+\frac{\gamma_i}{\mu_i}\right)-\frac{k_i'}{\varkappa k_i^2}\right]_\sigma \varkappa^2(a_{i0}^2+b_{i0}^2)k_i^2\times$$
$$\exp\left(-2\int_{s_0}^{\sigma}\Lambda_i ds\right)\left[1+\varkappa k_i(\sigma)\int_{s_0}^{\sigma}\Gamma_i(s)\exp\left(2\int_{\sigma}^{s}\Lambda_i ds_1\right)ds\right]^2=\chi$$

If the system parameters are the same and constant, i.e. there is no dependence on τ, then the problem admits of an explicit solution [3].

Suppose now that there are constraints on the control of the form $|u_i|\leqslant u_{i0}$ ($i=1,\ldots,n$). From the maximum condition on the Hamiltonian (1.7) we obtain $u_1^*=u_{i0}\operatorname{sign}(p_i\cos\psi-q_i\sin\psi)$. Averaging over ψ leads to the following expression for k_0

$$k_0=\frac{2}{\pi\nu}\sum_{i=1}^{n}\frac{u_{i0}}{\mu_i}\sqrt{\xi_i^2+\eta_i^2}-\frac{1}{2}\nu^{-1}\nu'(\alpha\xi+\beta\eta)+\nu^{-1}\mu^{-1}(\xi f_{0c}-\eta f_{0s})$$

Under the condition $\nu=\text{const}$ and $f_0\equiv 0$ we can obtain solutions for the fixed-time and the time-optimal problems since $\xi,\ \eta=\text{const}$, while α_i and β_i are piecewise-linear functions. The solution becomes complicated if there are singular controls. However, even in these situations the averaging method substantially simplifies the investigation of the optimal control problem.

3. Solution of the problem for a plane oscillator in polar coordinates. In certain problems it may be convenient to consider the equations of motion in a non-Cartesian coordinate system. For example, in the three-dimensional case (a spatial oscillator) the controls can be directed along the unit vectors of a spherical coordinate system, $u=(u_r,\ u_\theta,\ u_\varphi)$ or the perturbing forces are simply described in spherical coordinates. Let us write out the corresponding equations of motion:

$$mr^{\cdot\cdot}+kr-mr\theta^{\cdot 2}-mr\varphi^{\cdot 2}\sin^2\theta=\varepsilon u_r+\varepsilon f_r$$
$$0\leqslant r_1\leqslant r\leqslant r_2<\infty$$
$$mr\theta^{\cdot\cdot}+2mr^{\cdot}\theta^{\cdot}-mr\varphi^{\cdot 2}\sin\theta\cos\theta=\varepsilon u_\theta+\varepsilon f_\theta,\quad 0\leqslant\theta\leqslant\pi$$
$$mr\sin\theta\varphi^{\cdot\cdot}+2mr\varphi^{\cdot}\theta^{\cdot}\cos\theta+2mr^{\cdot}\varphi^{\cdot}\sin\theta=\varepsilon u_\varphi+\varepsilon f_\varphi$$
$$0\leqslant\varphi\leqslant 2\pi$$

To illustrate the arguments presented we investigate the plane problem ($\theta\simeq\pi/2$). The equations of motion of a plane oscillator in polar coordinates $(r,\ \varphi)$ under the control $\varepsilon u=(\varepsilon u_r,\ \varepsilon u_\varphi)$ are

$$\begin{aligned}
&r^{\cdot}=v_r, && r(t_0)=r_0 \\
&v_r^{\cdot}=v_\varphi^2/r-kr/m+\varepsilon u_r/m+\varepsilon\delta r^3/m, && v_r(t_0)=v_{r0} \\
&\varphi^{\cdot}=v_\varphi/r, && \varphi(t_0)=\varphi_0 \\
&v_\varphi^{\cdot}=-v_rv_\varphi/r+\varepsilon u_\varphi/m, && v_\varphi(t_0)=v_{\varphi 0}
\end{aligned}\qquad(3.1)$$

Here r is the distance to a fixed center, k is the rigidity factor of the restoring force, $\varepsilon\delta r^3$ is a nonlinear perturbation of the restoring force, m is the mass, εu_r and εu_φ are the radial and transversal components of the control vector. On the right in (3.1) we have given the corresponding initial conditions. For the sake of simplicity we assume that the problem's parameters are constant.

We reduce system (3.1) to the standard form with rotating phase [2, 3]. To do this we make use of the general solution of the unperturbed system, having taken the following set of integrals of motion:

$$1)\quad m(v_r^2+v_\varphi^2)/2+kr^2/2=E>0,\quad 2)\ mrv_\varphi=M,\quad 3)\ r^2=Ez/k$$

$$4)\quad \varphi=\gamma+\operatorname{arctg}\frac{\operatorname{tg}\psi+w}{\sqrt{1-w^2}}=\gamma+\operatorname{arctg}\left[w+\sqrt{1-w^2}\times \operatorname{tg}\left(\psi-\operatorname{arctg}\frac{w}{\sqrt{1-w^2}}\right)\right]$$

Here

$$z=1+w\sin 2\psi,\quad w=[1-(\nu M/E)^2]^{1/2}$$

$$\psi=\nu(t-t_0)+\psi_0,\quad \nu^2=k/m,\qquad E(1-w)/k\leqslant r^2\leqslant E(1+w)/k$$

The quantities E, M (w), ψ_0 and γ are constants of integration. We note that instead of relations (3) or (4) we can take the equivalent: $r^2=M^2/mE\,[1-w\sin 2(\varphi-\gamma)]$. Formulas (3) and (4) yield explicit expressions for $r(t)$ and $\varphi(t)$. The velocities corresponding to them are

$$r^\cdot=v_r(t)=w\sqrt{E/mz}\cos 2\psi,\quad v_\varphi(t)=r\varphi^\cdot=M/mr=(M/m)\sqrt{k/Ez}$$

By differentiating integrals (1)–(4) by virtue of the perturbed system (3.1), we obtain the required system of equations and initial values

$$\begin{aligned} E^\cdot &= \varepsilon v_r(u_r+\delta r^3)+\varepsilon v_\varphi u_\varphi, && E(t_0)=E_0 \\ M^\cdot &= \varepsilon r u_\varphi, && M(t_0)=M_0 \\ \gamma^\cdot &= -\frac{\sqrt{1-w^2}}{wz}\cos\psi(\cos\psi+w\sin\psi)\left(\frac{E^\cdot}{E}-\frac{M^\cdot}{M}\right), && \gamma(t_0)=\gamma_0 \\ \psi^\cdot &= \nu-\varepsilon\frac{w+\sin 2\psi}{2w\sqrt{mEz}}(u_r+\delta r^3)-\frac{M\nu\cos 2\psi}{2w\sqrt{mE^3z}}u_\varphi, && \psi(t_0)=\psi^\circ \end{aligned} \tag{3.2}$$

Here ν is a constant frequency; the initial conditions for the new variables are given in accord with (1)–(4)

$$2E_0=m(v_{r0}^2+v_{\varphi0}^2)+kr_0^2,\quad M_0=mr_0v_{\varphi0},\quad w_0=\sqrt{1-(\nu M_0/E_0)^2} \tag{3.3}$$

$$\sin 2\psi^\circ=(kr_0^2-E_0)/w_0,\quad \gamma_0=\varphi_0-\operatorname{arctg}\frac{\operatorname{tg}\psi^\circ+w_0}{\sqrt{1-w_0^2}}$$

ψ° and γ_0 are determined to within $N\pi$ (N is an integer). We note further, that the right-hand side of system (3.2) is periodic in ψ with period π, while the quantity w varies in accord with the equation

$$w^\cdot=\frac{\nu^2}{w}\left(\frac{M}{E}\right)^2\left(\frac{E^\cdot}{E}-\frac{M^\cdot}{M}\right) \tag{3.4}$$

In the investigations to be carried out we assume that the variable w lies within the limits: $w\in[w_1, w_2]$, where $w_1>0$, $w_2<1$, i.e. the motion is a nondegenerate ellipse if at each instant t we set ε equal to zero. Obviously, one of the first two equations in (3.2) can be replaced by Eq. (3.4) with the corresponding initial condition from (3.3). We note also that the right-hand side of system (3.2) is independent of γ. Therefore, if the functions u_r and u_φ are independent of γ, the equation for γ can be integrated separately, while the corresponding adjoint variable is retained.

We now pose the following "time-optimal with respect to energy" problem: transfer a system into a state with energy E_1 at a certain nonfixed instant t_1 in such a way that a functional of type (1.5)

$$J = \varepsilon l t_1 + \varepsilon \int_0^{t_1} (u_r^2 + u_\varphi^2)\, dt \tag{3.5}$$

is minimized. Here $l > 0$ is a given number. The remaining variables are assumed to be free at the right end. We write out the Hamiltonian (1.7)

$$H = -\varepsilon l - \varepsilon (u_r^2 + u_\varphi^2) + \varepsilon (p_E v_r + p_\gamma f_{\gamma r} + q f_{\psi r}) u_r + \varepsilon (p_E v_\varphi + p_M r + p_\gamma f_{\gamma\varphi} + q f_{\psi\varphi})\, u_\varphi + \nu q + \varepsilon p_E \delta r^3 v_r$$

Here p_E, p_M, p_γ and q are the corresponding adjoint variables, while the functions $f_{\gamma r}$, $f_{\gamma\varphi}$, $f_{\psi r}$ and $f_{\psi\varphi}$ are the coefficients of u_r and u_φ in the third and fourth equations (3.2), whose forms are immaterial for the present. From the condition (1.12) on H of maximum over u we obtain

$$u_r^* = \frac{1}{2}(p_E v_r + p_\gamma f_{\gamma r} + q f_{\psi r}) \quad \left(\frac{\partial H}{\partial u_r} = 0\right) \tag{3.6}$$

$$u_\varphi^* = \frac{1}{2}(p_E v_\varphi + p_M r + p_\gamma f_{\gamma\varphi} + q f_{\psi\varphi}) \quad \left(\frac{\partial H}{\partial u_\varphi} = 0\right)$$

We note the obvious properties of the resulting boundary-value problem. Since the function H^* is independent of γ, then $p_\gamma^\cdot = 0$, while as a result of the zero boundary condition we obtain that $p_\gamma \equiv 0$. Further, from the form of the right-hand side of the equation for q and from the zero boundary condition follows that $q = O(\varepsilon)$, i.e. we can neglect the quantity q in the first approximation being examined [2, 3]. Thus, averaging over ψ, we obtain

$$k_0 = \langle h \rangle = -l + \frac{\eta_E^2}{4}\frac{\xi}{m} + \frac{\eta_M^2}{4}\frac{\xi}{k} + \frac{\eta_E \eta_M}{2}\frac{\mu}{m}$$

since

$$\langle v_r^2 \rangle = \frac{\xi}{m}\left(1 - \frac{\mu}{\xi}\nu\right), \quad \langle r^2 \rangle = \frac{\xi}{k}, \quad \langle v_\varphi^2 \rangle = \frac{\mu}{m}\nu, \quad \langle v_\varphi r \rangle = \frac{\mu}{m}$$

Here ξ, η_E and μ, η_M are the averaged values of the variables E, p_E and M, p_M respectively, while the angle brackets signify averaging over ψ (see (1.15) and (1.18)). The averaged equations for ξ, η_E, μ, η_M are written in canonic form with the aid of the function k_0. The solution of the boundary-value problem is

$$\xi(s, s_1) = \left[\sqrt{E_0} - \frac{s}{s_1}\left(\sqrt{E_0} - \sqrt{E_1}\right)\right]^2 \tag{3.7}$$

$$\mu(s, s_1) = M_0 \frac{E_1}{E_0}\left[1 + \frac{s_1 - s}{s_1}\left(\sqrt{\frac{E_0}{E_1}} - 1\right)\right]^2$$

$$\eta_E = \frac{4m}{s_1}\left(\sqrt{\frac{E_0}{E_1}} - 1\right) \Big/ \left[1 + \frac{s_1 - s}{s_1}\left(\sqrt{\frac{E_0}{E_1}} - 1\right)\right]$$

$$\eta_M \equiv 0, \quad s_1 = \varepsilon t_1$$

An expression for s_1 is obtained by equating the quantity k_0 to zero; as a result

$$s_1 = 2\sqrt{\frac{m}{l}}\, \left|\sqrt{E_0} - \sqrt{E_1}\right| \tag{3.8}$$

According to (3.6) the approximate optimal control is

$$u_{r0}^* = \frac{\eta_E(s, s_1)}{2}\,\omega(s, s_1)\sqrt{\frac{\xi(s, s_1)}{m}}\cos 2\psi \Big/ \sqrt{1+\omega\sin 2\psi} \qquad (3.9)$$

$$u_{\varphi 0}^* = \frac{\eta_E(s, s_1)}{2m}\,\mu(s, s_1)\sqrt{\frac{k}{\xi(s, s_1)}}\Big/\sqrt{1+\omega\sin 2\psi}$$

$$\psi = \frac{\nu}{\varepsilon}s + \Delta\psi(s) + \psi^\circ + O(\varepsilon), \quad \omega = \sqrt{1-(\nu\mu/\xi)^2}$$

After ξ, η_E and μ, r_M have been found, expressions for the mean values of variables γ and ψ are obtained by quadrature. The approximate value of the slope angle α of the control vector u to the transversal is computed from (3.9)

$$\operatorname{tg}\alpha = u_{r0}^* / u_{\varphi 0}^* = \frac{\omega(s, s_1)\,\xi(s, s_1)}{\nu\mu(s, s_1)}\cos 2\psi$$

As a result, expressions (3.7)—(3.9) yield the approximate solution of the optimal control problem (3.2),(3.5) with the control process termination condition $E(t_1) = E_1$. Of course, this solution brings in an error of order ε in the slow variables and in the functional, while the process termination instant $t_1 = s_1/\varepsilon \sim 1/\varepsilon$ is determined with error $O(1)$. It is interesting to note that the values of ξ and μ do not depend explicitly on the rigidity k; they are also independent of δ, i.e. the nonlinear supplement of the restoring force. The change in frequency at the expense of this component equals $-3\delta\langle E\rangle/(4mk\nu)$ which corresponds to the one-dimensional case [2].

We note also that if $w \to 1$ $(w < 1)$, i.e. oscillations along a straight line, or $w \to 0$ $(w > 0)$, i.e. motion along a circle, then Eqs. (3.2) are degenerate and require additional investigation. In this case we can analyze the system's motion in a Cartesian coordinate system (see (1.1)) or in any other system, for example, in the system (a, b, ψ) (see Sects. 1 and 2), in which the equations of motion do not degenerate. In the Cartesian coordinate system the problem being examined is described, when $\delta = 0$, by a linear system of eight differential equations, and with the quadratic process termination condition $[(m/2)r'^2 + (k/2)r^2]_{t_1} = E_1$ for a specified value of t_1 can, in principle, be analytically solved exactly. However, for the determination of the control process termination instant t_1 we obtain a transcendental equation of type (1.11), admitting of an infinite set of roots as $\varepsilon \to 0$, among which it is necessary to choose the best in the sense of the criterion being analyzed. Since the resulting expressions are exceedingly cumbersome even for the one-dimensional case, the advantage of applying the averaging method for the investigation of the multi-dimensional systems is obvious [2, 3].

Let us now consider the problem of "pure time-optimality". Let $|u_r| \leqslant u_{r0}$, and $|u_\varphi| \leqslant u_{\varphi 0}$, and let the control process termination condition be $E(t_1) = E_1$, while the remaining phase variables are considered free at the right end. By computing, as before, the Hamiltonian (1.7) and retaining the same notation, from the condition of maximum of H over u we obtain

$$u_r^* = u_{r0}\operatorname{sign}(p_E v_r + p_\gamma f_{\gamma r} + q f_{\psi r}) \equiv \operatorname{sign} h_r$$
$$u_\varphi^* = u_{\varphi 0}\operatorname{sign}(p_E v_\varphi + p_M r + p_\gamma f_{\gamma\varphi} + q f_{\psi\varphi}) \equiv \operatorname{sign} h_\varphi$$

The system of equations and the boundary conditions for the adjoint variables have the form (1.8), where $h^* = |h_r| + |h_\varphi|$, $h^*(t_1) = 0$, $q(t) = -\varepsilon h^*/\nu$. In the first approximation the required slow variables are described by a canonic system of type

(1.14) with a Hamiltonian $k_0 = u_{r0} |\eta_E| \langle |v_r| \rangle + u_{\varphi 0} \langle | \eta_E v_\varphi + \eta_M r | \rangle$. From the equations we see that the best result is achieved under the condition $\operatorname{sign} \eta_E = \operatorname{sign} (\eta_E v_\varphi + \eta_M r) = \operatorname{sign} (E_1 - E_0)$. We then obtain the Cauchy problem

$$d\xi / ds = (u_{r0} \langle | v_r | \rangle + u_{\varphi 0} \langle v_\varphi \rangle) \operatorname{sign} (E_1 - E_0), \quad \xi (0) = E_0$$
$$d\mu / ds = u_{\varphi 0} \langle r \rangle \operatorname{sign} (E_1 - E_0), \quad \mu (0) = M_0$$

and the condition $\xi (s_1) = E_1$ determining s_1 uniquely. Now the required solution can be obtained by simple computational means.

In conclusion, we consider the case of variables $m (\tau)$ and $k (\tau)$ $(\delta = 0)$ and of constraints on the control of the form $u_x^2 + u_y^2 \leqslant u_0^2$. Let $\varepsilon\mu_x (\tau) u_x$ and $\varepsilon\mu_y (\tau) u_y$ be the controls, where μ_x and μ_y are specified functions. As a result of applying the procedures in Sect. 1 we obtain

$$u_x = u_0 (\mu_x / R)(\Pi_a \cos \psi - \Pi_b \sin \psi) + O (\varepsilon)$$
$$u_y = u_0 (\mu_y / R)(\Pi_c \cos \psi - \Pi_d \sin \psi) + O (\varepsilon)$$
$$R^2 = \mu_x^2 (\Pi_a \cos \psi - \Pi_b \sin \psi)^2 + \mu_y^2 (\Pi_c \cos \psi - \Pi_d \sin \psi)^2$$
$$\Pi (\tau) = \Pi (\sigma)[m (\tau)\nu (\tau) / m (\sigma)\nu (\sigma)]^{1/2}$$

($\Pi (\tau)$ is the averaged adjoint vector). The corresponding averaged phase vector is

$$\xi (\tau) = \xi_0 \left[\frac{m (\tau_0) \nu (\tau_0)}{m (\tau) \nu (\tau)}\right]^{1/2} + \frac{u_0}{\sqrt{m (\tau) \nu (\tau)}} \int_{s_0}^{s} \frac{\partial \langle R \rangle}{\partial \Pi} \frac{ds'}{\sqrt{m(\tau')\nu (\tau')}}$$
$$\xi_0 = (a_0, b_0, c_0, d_0)$$

$$\langle R (\tau, \Pi) \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} R (\tau, \Pi, \psi) d\psi = \frac{2}{\pi} \begin{cases} \sqrt{\beta}\, E \left(\sqrt{(\beta - \alpha)/\beta}\right), & \beta \geqslant \alpha \\ \sqrt{\alpha}\, E \left(\sqrt{(\alpha - \beta)/\alpha}\right), & \alpha > \beta \end{cases}$$
$$\alpha = (A + C)/2 + [(A - C)^2/4 + B^2]^{1/2}, \quad \beta = A + C - \alpha$$
$$A = \mu_x^2 \Pi_b^2 + \mu_y^2 \Pi_d^2, \quad B = - (\mu_x^2 \Pi_a \Pi_b + \mu_y^2 \Pi_c \Pi_d), \quad C = \mu_x^2 \Pi_a^2 + \mu_y^2 \Pi_c^2$$

Here E is the elliptic integral of the second kind, while the vector $\Pi (\sigma)$ is determined from the final expressions, namely, the transversality conditions at the right end.

REFERENCES

1. Chernous'ko, F. L., Some problems of optimal control with a small parameter. PMM Vol. 32, № 1, 1968.
2. Akulenko, L. D., Investigation of certain optimal systems by the averaging method. PMM Vol. 38, № 3, 1974.
3. Akulenko, L. D., Asymptotic solution of certain problems of time-optimal type. PMM Vol. 39, № 4, 1975.
4. Bogoliubov, N. N. and Mitropol'skii, Iu. A., Asymptotic Methods in the Theory of Nonlinear Oscillations. (English translation), Gordon and Breach, New York, 1964.
5. Volosov, V. M. and Morgunov, B. I., Averaging Method in the Theory of Nonlinear Oscillatory Systems. Moscow, Izd. Mosk. Univ., 1971.
6. Evtushenko, Iu. G., Approximate calculation of optimal control problems.

PMM Vol. 34, № 1, 1970.
7. Pontriagin, L. S., Boltianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., Mathematical Theory of Optimal Processes. (English translation), Pergamon Press, Book № 10176, 1964.
8. Grodzovskii, G. L., Ivanov, Iu. N. and Tokarev, V. V., Mechanics of Space Flight with Small Thrust. Moscow, "Nauka", 1966.
9. Samoilenko, A. M., Justification of the averaging principle for differential equations with a discontinuous right side. In: Approximate Methods for Solving Differential Equations. № 1, Kiev, Izd. Akad. Nauk UkrSSR, 1963.

Translated by N. H. C.

UDC 62-50

SMALL-PARAMETER METHOD FOR CONSTRUCTING APPROXIMATE STRATEGIES IN A CLASS OF DIFFERENTIAL GAMES

PMM Vol. 39, № 6, 1975, pp. 1006-1016
A. S. BRATUS'
(Moscow)
(Received November 12, 1974)

We examine a class of problems in which the pay-off is some function of the terminal state of a conflict-controlled system. When the opportunities of one of the players are small in relation with the opportunities of the other, we propose methods for constructing approximate optimal strategies of the players, based on solving the Bellman equation containing a small parameter. We have shown that the players' approximate optimal strategies can be constructed if the solutions of the corresponding optimal control problems are known. The error bounds for the methods are proved and examples are considered. The arguments used rely on the results in [1—6] on the theory of differential games and on [7—11] devoted to optimal control synthesis methods for systems subject to random perturbations of small intensity.

1. Statement of the problem. Let the motion of a conflict-controlled system be described by the nonlinear equation

$$\frac{dx}{dt} = F(x, t, u, v), \quad u \in P, \quad v \in Q_\varepsilon, \quad x[t_0] = x_0, \quad t \in [t_0, T] \tag{1.1}$$

Here x is an n-dimensional vector, u and v are r-dimensional control vectors of the first and second players, respectively, P and Q_ε are closed bounded sets, F is a continuous function satisfying a Lipschitz condition in x and v. The pay-off is the quantity $f[x(T)]$ determined at the terminal instant $t = T$ in the position $x(T)$ realized. The first player tries to minimize $f[x(T)]$ under the most unfavorable behavior of the second player. The second player's task is to guarantee the game's completion with the largest possible value of the pay-off. We assume that the opportunities of one of the players are small in comparison with the opportunities of the other. Namely, we assume that the set Q_ε can be contained within an r-dimensional sphere of radius ε small in relation to the minimal radius of the sphere which can contain set P. We